Commun. Math. Phys. (2003)
Digital Object Identifier (DOI) 10.1007/s00220-003-0972-8

Communications in
Mathematical Physics

Periodic Solutions of Nonlinear Wave Equations with General Nonlinearities*

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Received: 12 December 2002 / Accepted: 9 July 2003
Published online: 5 November 2003 –

Abstract: We present a variational principle for small amplitude periodic solutions, with fixed frequency, of a completely resonant nonlinear wave equation. Existence and multiplicity results follow by min-max variational arguments.

1. Introduction

We consider the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{1}$$

where $f(0) = f'(0) = 0$. We present a variational principle for finding small amplitude periodic solutions of (1), with *fixed* frequency, "branching off" from the *infinite dimensional* space of solutions of the linearized equation at 0,

$$u_{tt} - u_{xx} = 0, \qquad u(t, 0) = u(t, \pi) = 0. \tag{2}$$

Any solution $v = \sum_{j \geq 1} \xi_j \cos(jt + \theta_j) \sin(jx)$ of (2) is 2π-periodic in time.

For proving the existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome. The first (i) is a "small denominators" problem and the second (ii) is the presence of an infinite dimensional space of periodic solutions of the linearized equation with commensurable periods (infinitely many resonances among the linear frequencies of small oscillations): which solutions of the linearized equation (2) can be continued to solutions of the non linear equation (1)?

Problem (i) can be tackled via KAM techniques (see Kuksin [14] and references therein) or Nash-Moser Implicit Function Theorems (Craig-Wayne [11], Bourgain [9]).

* Supported by M.U.R.S.T. Variational Methods and Nonlinear Differential Equations.

Recently Bambusi [3], see also [5, 7], has introduced a strong non resonance condition for the frequency which allows to use, for second order equations, the standard Contraction mapping Theorem.

Concerning Problem (ii) no general results are yet available. In [5, 7] and [4] a method based on the Implicit Function Theorem, is introduced: non-degenerate critical points of a suitable functional can be continued, into periodic solutions of the nonlinear equation. Clearly the main difficulty in this method is to check, if ever true, such non-degeneracy condition (it is verified in [5, 6] for $f(u) = u^3 + o(u^3)$). In [12] some results are obtained for partially resonant cases, i.e. when only finitely many linear resonances are present, generalizing the variational principle of Weinstein [16] and Moser [15] for periodic solutions with fixed energy, see also [10]. The main drawback of this approach for infinite dimensional systems is that the frequency of the periodic solution appears as a "Lagrange multiplier" of a suitable action functional constrained on a sphere-like manifold and it is difficult to control these multipliers. On the other hand, for solving the small denominators problem (i), one has to fix the frequency in some good nonresonant set. Some control on the frequency is recovered requiring suitable non-degeneracy assumptions on the nonlinearity and this leads to existence results.

In this paper we introduce a *variational principle*, never used before for PDEs, for getting solutions with *fixed frequency*: we impose the frequency to satisfy a strong irrationality property for overcoming the small divisor problem (i) and we solve problem (ii) exploiting *min-max* variational arguments.

More precisely, by the classical Lyapunov-Schmidt procedure one has to solve two equations: the (P) equation, where the small denominators problem (i) appear, and the (Q) equation which is the "bifurcation equation" on the infinite dimensional space of solutions of (2), problem (ii). The (P) equation is solved by an Implicit Function Theorem. In order to focus our attention on problem (ii), we simplify the small denominators problem (i) imposing on the frequency ω the same strong irrationality condition as in [5] and using the Contraction mapping Theorem. Once the (P) equation is solved it remains the infinite dimensional (Q) equation: we solve it via a *variational principle* noting that it is the Euler-Lagrange equation of a "reduced action functional" Φ_ω defined in (7). Non-trivial critical points of Φ_ω can be obtained by min-max variational arguments. We underline that our variational approach for solving the (Q) equation keeps its validity if the (P) equation can be solved for more general frequencies with a Nash-Moser Implicit Function Theorem.

We illustrate in Sect. 3 the potentialities of our approach showing the existence of periodic solutions of Eq. (1) for the nonlinearities $f(u) = au^p$ $(a \neq 0)$, where p is either an *odd* or an *even* integer. Much sharper results can be obtained, see Sect. 4, and are proved in [8]: "optimal" multiplicity results, information on minimal periods and regularity of the solutions.

For finite dimensional Hamiltonian systems we quote the results of Fadell and Rabinowitz who proved in [13] existence and multiplicity of periodic solutions with fixed period through a minimax argument which exploits the natural S^1 symmetry of the problem (induced by time translations).

2. The Variational Principle

2.1. The variational Lyapunov-Schmidt reduction. We look for periodic solutions of (1) with frequency ω as $u(t, x) = q(\omega t, x)$, where $q(\cdot, x)$ is 2π periodic. After this normalization of the period we have to solve the problem

$$\omega^2 u_{tt} - u_{xx} + f(u) = 0, \quad u(t,0) = u(t,\pi) = 0, \quad u(t+2\pi, x) = u(t,x). \tag{3}$$

We look for solutions of (3) $u : \Omega \to \mathbf{R}$, where $\Omega := \mathbf{R}/2\pi\mathbf{Z} \times (0,\pi)$, in the Banach space

$$X := \left\{ u \in H^1(\Omega, \mathbf{R}) \cap L^\infty(\Omega, \mathbf{R}) \;\middle|\; u(t,0) = u(t,\pi) = 0,\ u(-t,x) = u(t,x) \right\}$$

endowed with the norm

$$||u||_X := |u|_\infty + ||u||,$$

$||\ ||$ being the norm on $H^1(\Omega)$ associated to the scalar product

$$(u,w) := \int_\Omega dt\, dx\, u_t w_t + u_x w_x.$$

We can restrict to the space X of functions even in time because Eq. (1) is reversible.

Any $u \in X$ can be developed in Fourier series as

$$u(t,x) = \sum_{l \geq 0, j \geq 1} u_{lj} \cos lt \sin jx,$$

and

$$(u,v) = \frac{\pi^2}{2} \sum_{l \geq 0, j \geq 1} u_{lj} w_{lj} (l^2 + j^2) \qquad \forall u, w \in X.$$

We denote by $|u|_{L^2} := (\int_\Omega |u|^2)^{1/2}$ the L^2-norm of $u \in X$ and by $(u,w)_{L^2} := \int_\Omega uw$ the L^2-scalar product.

The space of the solutions in $H_0^1(\Omega, \mathbf{R})$ of the linear equation $v_{tt} - v_{xx} = 0$ that are even in time is

$$V := \left\{ v(t,x) = \sum_{j \geq 1} \xi_j \cos(jt) \sin jx \;\middle|\; \xi_j \in \mathbf{R},\ \sum_{j \geq 1} j^2 \xi_j^2 < +\infty \right\}.$$

The space V can also be written as

$$V := \left\{ v(t,x) = \eta(t+x) - \eta(t-x) \;\middle|\; \eta(\cdot) \in H^1(\mathbf{T}),\ \eta \text{ odd} \right\},$$

where $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$.

On V the norm $||\ ||_X$ is equivalent to the H^1-norm $||\ ||$ since, for $v = \sum \xi_j \cos(jt) \sin(jx) \in V$, we have $|v|_\infty \leq \sum |\xi_j| \leq C||v||$ by the Cauchy-Schwarz inequality. Moreover the embedding $(V, ||\cdot||) \hookrightarrow (V, |\cdot|_\infty)$ is compact. For the sequel we endow V with the H^1 norm.

Remark 2.1. We do not use the same norm as in [5] on X because we want V to be endowed with the H^1 norm: it appears in the quadratic part at 0 of the reduced action functional Φ_ω, see (10). We do not work just in $H^1(\Omega)$ because Lemma 2.3 requires a space of bounded functions to work.

We next consider the *Lagrangian action functional* $\Psi : X \to \mathbf{R}$,

$$\Psi(u) := \int_0^{2\pi} dt \int_0^{\pi} dx \, \frac{\omega^2}{2} u_t^2 - \frac{1}{2} u_x^2 - F(u),$$

where $F(u) := \int_0^u f(s)\, ds$. It is easy to check that $\Psi \in C^1(X, \mathbf{R})$ and

$$D\Psi(u)[h] = \int_0^{2\pi} dt \int_0^{\pi} dx \, \omega^2 u_t h_t - u_x h_x - f(u)h, \quad \forall h \in X.$$

Critical points of Ψ are weak solutions of (3).

In order to find critical points of Ψ we perform a Lyapunov-Schmidt reduction. The space X can be decomposed as $X = V \oplus W$, where[1]

$$\begin{aligned} W &:= \Big\{ w \in X \;\Big|\; (w, v)_{L^2} = 0, \ \forall\, v \in V \Big\} \\ &= \Big\{ \sum_{l \geq 0, j \geq 1} w_{lj} \cos(lt) \sin jx \in X \;\Big|\; w_{jj} = 0 \, \forall j \geq 1 \Big\}. \end{aligned}$$

W is also the H^1-orthogonal of V in X.

Setting $u := v + w$ with $v \in V$ and $w \in W$, (3) is equivalent to the following system of two equations (called resp. the (Q) and the (P) equations)

$$\begin{aligned} &(Q) \qquad -\omega^2 v_{tt} + v_{xx} = \Pi_V f(v + w), \\ &(P) \qquad -\omega^2 w_{tt} + w_{xx} = \Pi_W f(v + w). \end{aligned}$$

Note that w solves the (P) equation (in a weak sense) if and only if it is a critical point of the restricted functional $w \to \Psi(v + w) \in \mathbf{R}$, *i.e.* if and only if

$$D\Psi(v + w)[h] = \int_0^{2\pi} dt \int_0^{\pi} dx \omega^2 w_t h_t - w_x h_x - f(v + w)h = 0, \quad \forall h \in W. \quad (4)$$

We first solve the (P) equation by an Implicit Function Theorem. Small denominators appear here. As already said, we will bypass this aspect of the problem assuming that $\omega \in \mathcal{W} := \cup_{\gamma > 0} \mathcal{W}_\gamma$, where $\mathcal{W}_\gamma$ is the set of strongly non-resonant frequencies introduced in [5]

$$\mathcal{W}_\gamma := \Big\{ \omega \in \mathbf{R} \;\Big|\; |\omega l - j| \geq \frac{\gamma}{l}, \quad \forall j \neq l \Big\}.$$

As proved in Remark 2.4 of [5], for $\gamma < 1/3$, the set $\mathcal{W}_\gamma$ is uncountable, has zero measure and accumulates to $\omega = 1$ both from the left and from the right.

Lemma 2.1. *For $\omega \in \mathcal{W}_\gamma$ the operator $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx} : D(L_\omega) \subset W \to W$ has a bounded inverse $L_\omega^{-1} : W \to W$ satisfying, for a positive constant C_1 independent of γ and ω, $||L_\omega^{-1}|| \leq (C_1/\gamma)$. Let $L^{-1} : W \to W$ be the inverse operator of $-\partial_{tt} + \partial_{xx}$. There exists $C_2 > 0$ such that $\forall r, s \in X$,*

$$\Big| \int_\Omega r(t, x) \Big(L_\omega^{-1} - L^{-1} \Big) (\Pi_W s(t, x)) \, dt \, dx \Big| \leq C_2 \frac{|\omega - 1|}{\gamma} ||r||_X \, ||s||_X. \quad (5)$$

[1] If $u = \sum u_{lj} \cos(lt) \sin(jx) \in X$, the H^1 norm, and so the L^∞ norm, of $\Pi_V(u) := \sum u_{jj} \cos(jt) \sin(jx)$ is finite and $\Pi_V(u) \in X$. As a consequence $\Pi_W(u) := u - \Pi_V(u) \in X$. Moreover the projectors $\Pi_V : X \to V$ and $\Pi_W : X \to W$ are continuous.

Proof. W.r.t. to Lemma 4.6 of [5] we have to prove that $L_\omega^{-1} W \subset L^\infty(\Omega)$ and (5). It is done in the Appendix. □

Fixed points of the nonlinear operator $\mathcal{G} : W \to W$ defined by $\mathcal{G}(w) := L_\omega^{-1} \Pi_W f(v + w)$ are solutions of the (P) equation. In order to prove that $\mathcal{G}$ is a contraction for v sufficiently small, we need the following standard lemma on the Nemytsky operator induced by f.

Lemma 2.2. *The Nemitski operator $u \to f(u)$ is in $C^1(X, X)$ and its derivative at u is $Df(u)[h] = f'(u)h$. Moreover, if $f(0) = f'(0) = \ldots = f^{(p-1)}(0) = 0$, there is $\rho_0 > 0$ and positive constants C_3, C_4, depending only on f, such that, $\forall u \in X$ with $||u||_X < \rho_0$,*

$$||f(u)||_X \le C_3 |u|_\infty^{p-1} ||u||_X \le C_3 ||u||_X^p, \quad ||f'(u)h||_X \le C_4 |u|_\infty^{p-2} ||u||_X \, ||h||_X \le C_4 ||u||_X^{p-1} ||h||_X. \tag{6}$$

The next lemma, based on the Contraction mapping Theorem, is similar to Lemma 4.6 of [5].

Lemma 2.3. *Let $f(0) = f'(0) = \ldots = f^{(p-1)}(0) = 0$ and assume that $\omega \in \mathcal{W}_\gamma$. There exists $\rho > 0$ such that, $\forall v \in \mathcal{D} := \{v \in V \mid ||v||^{p-1}/\gamma < \rho\}$ there exists a unique $w(v) \in W$ which solves the (P) equation. Moreover*

- *$i)$ $||w(v)||_X = O(||v||^p/\gamma)$;*
- *$ii)$ $\forall r \in X$, $|\int_\Omega (w(v) - L_\omega^{-1} \Pi_W f(v))\, r| = O\Big((1/\gamma^2) ||v||^{2p-1} ||r||_X\Big)$;*
- *$iii)$ the map $v \to w(v)$ is in $C^1(\mathcal{D}, W)$;*
- *$iv)$ $w(-v)(t, x) = w(v)(t + \pi, \pi - x)$.*

Once the (P) equation is solved by $w(v) \in W$, there remains the infinite dimensional (Q) equation $-\omega^2 v_{tt} + v_{xx} = \Pi_V f(v + w(v))$. We claim that such an equation is the Euler-Lagrange equation of the *reduced Lagrangian action functional* $\Phi_\omega : \mathcal{D} \to \mathbf{R}$ defined by

$$\Phi_\omega(v) := \Psi(v + w(v)) = \int_0^{2\pi} dt \int_0^\pi dx \, \frac{\omega^2}{2}(v_t + (w(v))_t)^2 - \frac{1}{2}(v_x + (w(v))_x)^2 - F(v + w(v)). \tag{7}$$

Indeed, by Lemma 2.3-$iii)$ Φ_ω is in $C^1(\mathcal{D}, \mathbf{R})$ and $\forall h \in V$,

$$D\Phi_\omega(v)[h] = D\Psi(v + w(v))\Big[h + dw(v)[h]\Big] = D\Psi(v + w(v))[h], \tag{8}$$

since $dw(v)h \in W$ and $w(v)$ solves the (P) equation (recall (4)). By (8), for $v, h \in V$,

$$D\Phi_\omega(v)[h] = \int_\Omega dt\, dx\, \omega^2 v_t h_t - v_x h_x - \Pi_V f(v + w(v))h = \varepsilon(v, h) - \int_\Omega dt\, dx\, \Pi_V f(v + w(v))h, \tag{9}$$

where $\varepsilon := (\omega^2 - 1)/2$. By (9), a critical point $v \in \mathcal{D} \subset V$ of Φ_ω is a solution of the (Q)-equation:

Theorem 2.1. *If $v \in \mathcal{D} \subset V$ is a critical point of the reduced action functional $\Phi_\omega : \mathcal{D} \to \mathbf{R}$ then $u = v + w(v)$ is a weak solution of (3).*

Remark 2.2. This reduction gives also a necessary condition: any critical point u of Φ_ω, sufficiently close to 0, can be written as $u = v + w$, where $v \in V$ is a critical point of Φ_ω and $w = w(v)$, see [1].

We have reduced the problem of finding non-trivial solutions of the infinite dimensional (Q)-equation to the problem of finding non trivial critical points of the reduced action functional Φ_ω (by (9) $v = 0$ is a critical point of Φ_ω for all ω).

By (7) and formula (4) with $h = w(v)$, the reduced action functional can be developed as

$$\Phi_\omega(v) = \int_\Omega dt\, dx\, \frac{\omega^2}{2} v_t^2 - \frac{v_x^2}{2} - F(v + w(v)) + \frac{1}{2} f(v + w(v)) w(v),$$

and since $|v_t|_{L^2}^2 = |v_x|_{L^2}^2 = ||v||^2/2$,

$$\Phi_\omega(v) = \frac{\varepsilon}{2}||v||^2 + \int_\Omega dt\, dx \left[\frac{1}{2} f(v + w(v)) w(v) - F(v + w(v))\right], \tag{10}$$

where $\varepsilon := (\omega^2 - 1)/2$.

By (10) and the bounds of Lemma 2.3, Φ_ω possesses a local minimum at the origin and we shall show that Φ_ω satisfies the geometrical hypotheses of the Mountain Pass Theorem [2]. However we can not apply directly this theorem since Φ_ω is defined only on a neighborhood of the origin. In the next Subsect. 2.2, we prove an abstract theorem which can be applied directly to our problem.

2.2. Existence of critical points: An abstract result. Let $\Phi : B_r \subset E \to \mathbf{R}$ be a C^1 functional defined on the ball $B_r := \{v \in E \mid ||v|| < r\}$ of a Hilbert space E with scalar product $(\cdot, \cdot)$, of the form

$$\Phi(v) = \frac{\varepsilon}{2}||v||^2 - G(v) + R(v), \tag{11}$$

where $G \not\equiv 0$ and

- $(H1)$ $G \in C^1(E, \mathbf{R})$ is homogeneous of degree $q + 1$ with $q > 1$, i.e. $G(\lambda v) = \lambda^{q+1} G(v)\ \forall \lambda \in \mathbf{R}_+$;
- $(H2)$ $\nabla G : E \to E$ is compact;
- $(H3)$ $R \in C^1(B_r, \mathbf{R})$, $R(0) = 0$ and for any $r' \in (0, r)$, ∇R maps $B_{r'}$ into a compact subset of E.

Theorem 2.2. *Let G satisfy $(H1)$, $(H2)$ and suppose that $G(v) > 0$ for some $v \in E$ (resp. $G(v) < 0$). There is $\alpha > 0$ (depending only on G) and $\varepsilon_0 > 0$ (depending on r) such that, for all $R \in C^1(B_r, \mathbf{R})$ satisfying $(H3)$ and*

$$|(\nabla R(v), v)| \le \alpha ||v||^{q+1}, \forall v \in B_r \tag{12}$$

for all $\varepsilon \in (0, \varepsilon_0)$ (resp. $\in (-\varepsilon_0, 0)$), Φ has a non-trivial critical point $v \in B_r$ satisfying $||v|| = O(\varepsilon^{1/(q-1)})$.

Remark 2.3. The key difference with the approach of [5] is the following: instead of trying to show that the functional $(\varepsilon/2)||v||^2 - G(v)$ possesses non-degenerate critical points (if ever true), and then continuing them through the Implicit Function Theorem as critical points of Φ, we find critical points of Φ showing that the nonlinear perturbation term R does not affect the mountain pass geometry of the functional $(\varepsilon/2)||v||^2 - G(v)$. Actually, in [5], non-degenerate critical points of G constrained on S are continued. This is equivalent to what was said before since any critical point $\widetilde{v}$ of G constrained to S gives rise, by homogeneity, to a critical point $(\varepsilon/(q+1)G(\widetilde{v}))^{1/q-1}\widetilde{v} \in E$ of the functional $(\varepsilon/2)||v||^2 - G(v)$, provided that ε and $G(\widetilde{v})$ have the same sign.

Proof of Theorem 2.2. For definiteness we assume that $G(v) > 0$ for some $v \in V$ and we take $\varepsilon > 0$.

The steps of the proof are the following: 1) Define on the whole space E a new functional $\widetilde{\Phi}$ which is an extension of $\Phi_{|V}$ for some neighborhood V of 0, in such a way that $\widetilde{\Phi}$ possesses the Mountain-Pass geometry, see (15). 2) Derive by the Mountain-Pass Theorem the existence of a "Palais-Smale" sequence for $\widetilde{\Phi}$, see (16). 3) Prove that (v_n) converges to some critical point v in an open ball where $\widetilde{\Phi}$ and Φ coincide.

Step 1. Let us consider $\overline{v} \in S := \{v \in E \mid ||v|| = 1\}$ such that $m := G(\overline{v}) > 0$. The function $t \to (\varepsilon/2)||t\overline{v}||^2 - G(t\overline{v}) = (\varepsilon/2)t^2 - t^{q+1}m$ attains its maximum at

$$r_\varepsilon := \left(\frac{\varepsilon}{(q+1)m}\right)^{1/(q-1)}$$

with maximum value $((1/2) - 1/(q+1))\varepsilon r_\varepsilon^2$.

Let $\lambda = [0, +\infty) \to \mathbf{R}$ be a smooth cut-off non-increasing function such that

$$\lambda(s) = 1 \ \text{ if } \ s \in [0, 4] \quad \text{and} \quad \lambda(s) = 0 \ \text{ if } \ s \in [16, +\infty). \tag{13}$$

Choose $\varepsilon_0 > 0$ small enough, such that $4r_{\varepsilon_0} < r$. For all $0 < \varepsilon < \varepsilon_0$
we define a functional $\widetilde{R}_\varepsilon : E \to \mathbf{R}$ by

$$\widetilde{R}_\varepsilon(v) := \lambda\left(\frac{||v||^2}{r_\varepsilon^2}\right) R(v) \ \text{ if } \ v \in B_r \qquad \text{and} \qquad \widetilde{R}_\varepsilon(v) := 0 \ \text{ if } \ v \notin B_r.$$

$\widetilde{R}_\varepsilon \in C^1(E, \mathbf{R})$, $\widetilde{R}_{\epsilon|B_{2r_\epsilon}} = R_{|B_{2r_\epsilon}}$ and, by (12)-(13), there is a constant C depending on λ and q only, such that

$$\forall v \in E \quad |\widetilde{R}_\varepsilon(v)| \le C\alpha||v||^{q+1} \qquad \text{and} \qquad |(\nabla\widetilde{R}_\varepsilon(v), v)| + |\widetilde{R}_\varepsilon(v)| \le C\alpha r_\varepsilon^{q+1}. \tag{14}$$

Then we can define $\widetilde{\Phi}$ on the whole E as

$$\widetilde{\Phi}(v) := \frac{\varepsilon}{2}||v||^2 - G(v) + \widetilde{R}_\varepsilon(v).$$

$\widetilde{\Phi}(0) = 0$ and $\widetilde{\Phi}$ possesses the Mountain pass geometry: $\exists\delta > 0$ and $w \in \mathbf{R}\overline{v}$ with $||w|| > \delta$, such that

$$(i) \ \inf_{v \in \partial B_\delta} \widetilde{\Phi}(v) > 0, \qquad (ii) \ \widetilde{\Phi}(w) < 0. \tag{15}$$

Equation (15)-(ii) holds for $w = R\overline{v}$ with R large enough since $\lim_{R\to\infty} \widetilde{\Phi}(R\overline{v}) = -\infty$. For Eq. (15)-($i$), observe first that by the compactness of $\nabla G : E \to E$, G maps the sphere S into a bounded set. Hence there is a constant $K > 0$ such that $|G(v)| \le K||v||^{q+1}$, and by (14), any $\delta \in (0, R)$ such that $(\varepsilon/2)\delta^2 - (K + C\alpha)\delta^{q+1} > 0$ is suitable.

Step 2. Define the Mountain pass paths

$$\Gamma = \left\{\gamma \in C([0,1], E) \;\middle|\; \gamma(0) = 0,\ \gamma(1) = w\right\}$$

and the Mountain-pass level

$$c_\varepsilon = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \widetilde{\Phi}(\gamma(s)).$$

By (15)-(i), $c_\varepsilon > 0$. By the Mountain-Pass Theorem [2] there exists a sequence (v_n) such that

$$\nabla\widetilde{\Phi}(v_n) \to 0, \qquad \widetilde{\Phi}(v_n) \to c_\varepsilon, \tag{16}$$

(Palais-Smale sequence).

Step 3. We shall prove that for n large enough v_n lies in a ball B_h for some $h < 2r_\varepsilon$. For this we need an estimate of the level c_ε. By the definition of c_ε and (14),

$$c_\varepsilon \le \max_{s\in[0,1]} \widetilde{\Phi}(sR\overline{v}) \le \max_{t\in[0,R]} \frac{\varepsilon}{2}||t\overline{v}||^2 - (m - C\alpha)||t\overline{v}||^{q+1}.$$

Computing the maximum in the right-hand side, we find the estimate

$$c_\varepsilon \le \varepsilon\left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\frac{m}{m - C\alpha}\right)^{2/(q-1)} r_\varepsilon^2. \tag{17}$$

We claim that $\limsup_{n\to+\infty} ||v_n|| < 2r_\varepsilon$. If not, up to a subsequence, $\lim_{n\to\infty} ||v_n|| := \nu \in [2r_\varepsilon, +\infty]$. By the homogeneity of G, $(\nabla\widetilde{\Phi}(v), v) = \varepsilon||v||^2 - (q+1)G(v) + (\nabla\widetilde{R}_\varepsilon(v), v)$ and, by (16),

$$(\nabla\widetilde{\Phi}(v_n), v_n) = \varepsilon||v_n||^2 - (q+1)G(v_n) + (\nabla\widetilde{R}_\varepsilon(v_n), v_n) = \mu_n||v_n||$$

with $\lim_{n\to\infty} \mu_n = 0$. This implies

$$\varepsilon\left(\frac{1}{2} - \frac{1}{q+1}\right)||v_n||^2 = \widetilde{\Phi}(v_n) - \frac{\mu_n}{q+1}||v_n|| - \widetilde{R}_\varepsilon(v_n) + \frac{1}{q+1}(\nabla\widetilde{R}_\varepsilon(v_n), v_n).$$

Since $\widetilde{\Phi}(v_n) \to c_\varepsilon$ and using (14), we derive that the sequence $(||v_n||)$ is bounded and so $\nu < \infty$. Taking limits as $n \to \infty$, we obtain

$$\varepsilon\Big(\frac{1}{2} - \frac{1}{q+1}\Big)\nu^2 \le c_\varepsilon + C\alpha r_\varepsilon^{q+1} \le c_\varepsilon + C'\frac{\alpha}{m}\varepsilon r_\varepsilon^2, \tag{18}$$

by the definition of r_ε and for some positive constant C'. Since $\nu \ge 2r_\varepsilon$, (18) contradicts estimate (17), provided α has been chosen small enough (depending on q and m only).

Thus $v_n \in B_h$ (for n large), for some $h < 2r_\varepsilon$, and since $\Phi \equiv \widetilde{\Phi}$ on B_{2r_ε},

$$\nabla\widetilde{\Phi}(v_n) = \nabla\Phi(v_n) = \varepsilon v_n - \nabla G(v_n) + \nabla R(v_n) \to 0.$$

Since (v_n) is bounded, by the compactness assumptions ($H2$) and ($H3$), (v_n) converges in B_{2r_ε} to some non-trivial critical point v of Φ at the critical level $c_\varepsilon > 0$. □

To complete this section, we note that when Φ is invariant under some symmetry group (e.g. Φ is even), multiplicity of non-trivial critical points can be obtained, as in the symmetric version of the Mountain pass Theorem [2].

We remark that the reduced action functional Φ_ω defined in Subsect. 2.1 is even[2]. Indeed defining the linear operator $\mathcal{I} : X \to X$ by $(\mathcal{I}u)(t,x) := u(t+\pi, \pi - x)$, $\Psi \circ \mathcal{I} = \Psi$, and, by Lemma 2.3-$iv$) and since $-v = \mathcal{I}v$,

$$\Phi_\omega(-v) = \Psi(-v + w(-v)) = \Psi(\mathcal{I}(v + w(v))) = \Psi(v + w(v)) = \Phi_\omega(v). \quad (19)$$

We can prove that the number N_ω of non-trivial critical points of Φ_ω in $\mathcal{D}$ increases to $+\infty$ as the frequency ω tends to 1. These and other results are presented in Sect. 4 and proved in [8].

3. Applications to Nonlinear Wave Equations

As an illustration of our method we prove existence of periodic solutions of the nonlinear wave equation (1) when $f(u) = au^p$ ($a \neq 0$) for $p \geq 2$ odd and p even integer. Here $F(u) := au^{p+1}/(p+1)$.

3.1. Case I: p odd.

Lemma 3.1. *Let $f(u) = au^p$ for an odd integer p. Then the reduced action functional $\Phi_\omega : \mathcal{D} \to \mathbf{R}$ defined in (7) has the form (11) with $\varepsilon = (\omega^2 - 1)/2$,*

$$G(v) := \int_\Omega F(v) = a\int_\Omega \frac{v^{p+1}}{p+1} \quad \text{and} \quad R(v) := \int_\Omega -F(v + w(v)) + F(v) + \frac{1}{2} f(v + w(v))w(v).$$

Moreover $(\nabla R(v), v) = O(||v||^{2p})$.

Proof. We find, by (9), $(\nabla R(v), v) = \int_\Omega (f(v) - f(v + w(v)))v$ and so, by Lemma 2.3 and $|v|_\infty \leq C||v||$,

$$|(\nabla R(v), v)| \leq \int_\Omega |f(v) - f(v + w(v)|\, |v| \leq |w(v)|_\infty |v|_\infty^p = O(||v||^{2p}). \quad \square$$

We have to check the compactness properties $(H2)$ and $(H3)$.

Lemma 3.2. *G and R satisfy assumptions* $(H2)$ *and* $(H3)$.

Proof. We have $(\nabla G(v), h) = \int_\Omega av^p h$ and $(H2)$ stems from the compactness of the embedding $H^1(\Omega) \hookrightarrow L^{2p}(\Omega)$. Now let (v_k) be some sequence in $B_{r'}$, with $r' < r$, $B_r \subset \mathcal{D}$. Then, up to a subsequence, $v_k \rightharpoonup \overline{v} \in V$ weakly for the H^1 topology and $v_k \to \overline{v}$ in $|\cdot|_\infty$. Moreover since $w_k := w(v_k)$ too is bounded we can also assume that $w_k \rightharpoonup \overline{w}$ weakly in H^1 and $w_k \to \overline{w}$ in $|\cdot|_{L^q}$ norm for all $q < \infty$. We claim that $\nabla R(v_k) \to \overline{R}$, where $(\overline{R}, h) = \int_\Omega f(\overline{v} + \overline{w})h - a\overline{v}^p h$. Indeed, since $w_k \to \overline{w}$ in $|\cdot|_{L^q}$, it converges (up to a subsequence) also $a.e.$ We can deduce, by the Lebesgue dominated convergence theorem, that $f(v_k + w_k) \to f(\overline{v} + \overline{w})$ in L^2 since $f(v_k + w_k) \to f(\overline{v} + \overline{w})$ a.e. and $(f(v_k + w_k))$ is bounded in L^∞. Hence, since $(\nabla R(v_k), h) = \int_\Omega f(v_k + w_k)h - av_k^p h$, $\nabla R(v_k) \to \overline{R}$. $\square$

[2] Not restricting to the space X of functions even in time the reduced functional would inherit the natural S^1 invariance symmetry defined by time translations.

$G(v)$ is homogeneous of order $p+1$ and for $a > 0$, $G(v) := a \int_\Omega v^{p+1} > 0 \ \forall v \neq 0$. By Lemma 3.1 we can choose r small enough so that $\forall v \in B_r \ (\nabla R(v), v) \leq \alpha ||v||^{p+1}$, where α is defined in (12). Applying Theorem 2.2 we obtain :

Theorem 3.1. *Let $f(u) = au^p$ ($a \neq 0$) for an odd integer $p \geq 3$. There exists a positive constant $C_5 := C_5(f)$ such that, $\forall \omega \in \mathcal{W}_\gamma$ satisfying $|\omega - 1| \leq C_5$ and $\omega > 1$ if $a > 0$ (resp. $\omega < 1$ if $a < 0$), Eq. (1) possesses at least one $2\pi/\omega$-periodic, even in time solution.*

3.2. Case II: p even. The case $f(u) = au^p$ with p even integer requires more attention since, by Lemma 3.4 below, $\int_\Omega v^{p+1} \equiv 0$.

Lemma 3.3. *Let $m : \mathbf{R}^2 \to \mathbf{R}$ be 2π-periodic w.r.t. both variables. Then*

$$\int_0^{2\pi} \int_0^{\pi} m(t+x, t-x)\, dt\, dx = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} m(s_1, s_2)\, ds_1\, ds_2.$$

Proof. Make the change of variables $(s_1, s_2) = (t+x, t-x)$ and use the periodicity of m. □

Lemma 3.4. *If $v \in V$ then $v^p \in W$. In particular $\int_\Omega v^{p+1} = 0$.*

Proof. For all $v(t,x) = \eta(t+x) - \eta(t-x)$, $u(t,x) = q(t+x) - q(t-x) \in V$, by Lemma 3.3,

$$\int_0^{2\pi} \int_0^{\pi} v^{2p}(t,x) u(t,x)\, dx\, dt$$
$$= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \Big(\eta(s_1) - \eta(s_2)\Big)^{2p} (q(s_1) - q(s_2))\, ds_1\, ds_2 = 0,$$

because $(s_1, s_2) \mapsto (\eta(s_1) - \eta(s_2))^{2p}(q(s_1) - q(s_2))$ is an odd function. □

We have to look for the dominant nonquadratic term in the reduced functional Φ_ω.

Lemma 3.5. *Let $f(u) = au^p$ with $p \geq 2$ an even integer. Then $\Phi_\omega : \mathcal{D} \to \mathbf{R}$ defined in (7) has the form (11) with $\varepsilon := (\omega^2 - 1)/2$,*

$$G(v) := \frac{a^2}{2} \int_\Omega v^p L^{-1} v^p \quad \text{and}$$
$$R(v) := \int_\Omega \frac{1}{2} f(v + w(v)) w(v) - F(v + w(v)) - \frac{a^2}{2} v^p L^{-1} v^p.$$

Moreover $(\nabla R(v), v) = O(||v||^{3p-1} + |\varepsilon|\, ||v||^{2p})$.

Proof. We find, by (9), $(\nabla R(v), v) = \int_\Omega f(v + w(v))v - pa^2 v^p L^{-1} v^p$. Developing in Taylor series, using Lemma 2.3-i)-ii) and $\int_\Omega v^{p+1} = 0$, we obtain

$$(\nabla R(v), v) = \int_\Omega f(v)v + f'(v)v\, w(v) + O(||v||^{3p-1}) - pa^2 v^p L^{-1} v^p$$
$$= \int_\Omega pa^2 v^p L_\omega^{-1} v^p + O(||v||^{3p-1}) - pa^2 v^p L^{-1} v^p$$

$$= \int_\Omega pa^2 v^p \Big[L_\omega^{-1} v^p - L^{-1} v^p \Big] + O(||v||^{3p-1})$$
$$= O(|\varepsilon| \, ||v||^{2p} + ||v||^{3p-1}),$$

by (5). □

The next lemma, proved in the Appendix, ensures that $G \not\equiv 0$.

Lemma 3.6. $G(v) := (a^2/2) \int_\Omega v^p L^{-1} v^p < 0, \ \forall v \neq 0.$

G is homogeneous of degree $2p$ (property $(H1)$); ∇G, ∇R still satisfy assumptions $(H2)$ and $(H3)$. By Lemma 3.5, choosing first r then ε small enough, we can apply Theorem 2.2 and we get:

Theorem 3.2. *Let $f(u) = au^p$ $(a \neq 0)$ and p be an even integer. There is $C_6 := C_6(f) > 0$ such that, $\forall \omega \in \mathcal{W}_\gamma$, $\omega < 1$, with $|\omega - 1| \leq C_6$, Eq. (1) possesses at least one $2\pi/\omega$-periodic, even in time solution.*

4. Further Results

Much stronger results than Theorems 3.1 and 3.2 can be obtained. For any smooth nonlinearity $f(u) = O(u^2)$ with some $f^p(0) \neq 0$, for $\omega \in \mathcal{W}$ in a right or left neighborhood of 1, we prove in [8] the existence of a *large* number N_ω of $2\pi/\omega$-periodic *classical* $C^2(\Omega)$ solutions $u_1, \ldots, u_n, \ldots, u_{N_\omega}$ with $N_\omega \to +\infty$ as $\omega \to 1$ ($N_\omega \approx \sqrt{\gamma^\tau / |\omega - 1|}$ for some $\tau \in [1, 2]$). Moreover the *minimal period* of the n^{th} solution u_n is proved to be $2\pi/n\omega$.

The following theorems are proved in [8]. For $\omega \in \mathcal{W} := \cup_{\gamma > 0} \mathcal{W}_\gamma$ define $\gamma_\omega := \max\{\gamma \mid \omega \in \mathcal{W}_\gamma\}$.

Theorem 4.1. *Let $f(u) = au^p + \text{h.o.t.}$ $(a \neq 0)$ for an odd integer $p \geq 3$. Then there exists a positive constant $C_7 := C_7(f)$ such that, $\forall \omega \in \mathcal{W}$ and $\forall n \in \mathbf{N} \backslash \{0\}$ satisfying*

$$\frac{|\omega - 1| n^2}{\gamma_\omega} \leq C_7$$

and $\omega > 1$ if $a > 0$ (resp. $\omega < 1$ if $a < 0$), Eq. (1) possesses at least one pair of even periodic in time classical C^2 solutions with minimal period $2\pi/(n\omega)$.

Theorem 4.2. *Let $f(u) = au^p$ $(a \neq 0)$ for some even integer p. Then there exists a positive constant C_8 depending only on f such that, $\forall \omega \in \mathcal{W}$ with $\omega < 1$, $\forall n \geq 2$ such that*

$$\frac{(|\omega - 1| n^2)^{1/2}}{\gamma_\omega} \leq C_8,$$

Equation (1) possesses at least one pair of even periodic in time classical C^2 solutions with minimal period $2\pi/(n\omega)$. If $p = 2$ the existence result holds true for $n = 1$ as well.

When $f(u) = au^p + o(u^p)$, p even, two other cases have to be considered, according to the behaviour of the higher order terms of the nonlinearity f. We shall not enter into the details, see [8].

5. Appendix

Proof of Lemma 2.1. Writing $w(t,x) = \sum_{l\geq 0, j\geq 1, j\neq l} w_{l,j}\cos(lt)\sin(jx) \in W$ we claim that

$$|L_\omega^{-1}w|_\infty \leq \sum_{l\geq 0, j\geq 1, j\neq l} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)} =: S \leq C\Big(|w|_{L^2} + \frac{1}{\gamma}||w||\Big) \leq \frac{C}{\gamma}||w||.$$

For $l \in \mathbf{N}$, let $e(l) \in \mathbf{N}$ be defined by $|e(l) - \omega l| = \min_{j\in\mathbf{N}} |j - \omega l|$. Since ω is not rational, $e(l)$ is the only integer e such that $|e - \omega l| < 1/2$. $S = S_1 + S_2$, where

$$S_1 := \sum_{l\geq 0, j\geq 1, j\neq l, j\neq e(l)} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)} \text{ and } \quad S_2 := \sum_{l\geq 0, e(l)\neq l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))}.$$

We first find an upper bound for S_1. For $j \neq e(l)$ we have that $|j - \omega l| \geq |j - e(l)| - |e(l) - \omega l| \geq |j - e(l)| - 1/2 \geq |j - e(l)|/2$. Moreover, since $|e(l) - \omega l| < 1/2$, it is easy to see (remember that $\omega \geq 1/2$) that $e(l) + l \leq 4\omega l$, and hence $|j - e(l)| + l \leq j + e(l) + l \leq 4(j + \omega l)$. Defining $w_{l,j}$ by $w_{l,j} = 0$ if $j \leq 0$ or $j = l$, we deduce

$$S_1 \leq \sum_{l\geq 0, j\in\mathbf{Z}, j\neq e(l)} \frac{8|w_{l,j}|}{|j - e(l)|(|j - e(l)| + l)}.$$

Hence, by the Cauchy-Schwarz inequality, $S_1 \leq 8R_1|w|_{L^2}$, where

$$R_1^2 = \sum_{l\geq 0, j\in\mathbf{Z}, j\neq e(l)} \frac{1}{(j - e(l))^2(|j - e(l)| + l)^2} = \sum_{l\geq 0, j\in\mathbf{Z}, j\neq 0} \frac{1}{j^2(|j| + l)^2}$$
$$\leq \sum_{l\geq 0, j\in\mathbf{Z}, j\neq 0} \frac{1}{j^2(1 + l)^2} < \infty.$$

We now find an upper bound for S_2. Since $\omega \in \mathcal{W}_\gamma$, for $l \neq e(l)$, $|\omega l - e(l)||\omega l + e(l)| \geq \gamma l^{-1}(\omega l + e(l)) \geq \gamma$. Hence, still by the Cauchy-Schwarz inequality, $S_2 \leq (1/\gamma)\sum_{e(l)\neq l} |w_{l,e(l)}| \leq (C/\gamma)||w||$.

Finally we prove (5). Writing $r = \sum_{l\geq 0, j\geq 1} r_{lj}\cos(lt)\sin(jx)$, $s = \sum_{l\geq 0, j\geq 1} s_{lj}\cos(lt)\sin(jx)$,

$$L^{-1}\Pi_W s = \sum_{j\neq l} \frac{s_{lj}}{l^2 - j^2}\cos(lt)\sin(jx)\ , \qquad L_\omega^{-1}s = \sum_{j\neq l} \frac{s_{lj}}{\omega^2 l^2 - j^2}\cos(lt)\sin(jx),$$

$$\int_\Omega r(L_\omega^{-1} - L^{-1})(\Pi_W s)\, dt\, dx = \pi^2 \sum_{j\neq l} \frac{s_{lj}r_{lj}\ (1 - \omega^2)l^2}{(\omega^2 l^2 - j^2)(l^2 - j^2)}. \tag{20}$$

By (20), since $\omega^2 l^2 - j^2 \geq \omega\gamma$ and $l^2 - j^2 \geq 1$,

$$\Big|\int_\Omega r(L_\omega^{-1} - L^{-1})(\Pi_W s)\, dt\, dx\Big| \leq C\frac{|\omega - 1|}{\gamma}\sum_{j\neq l} |s_{l,j}||r_{l,j}|\, l^2 \leq C'\frac{|\omega - 1|}{\gamma}||r||_X\, ||s||_X,$$

which proves (5). $\square$

Proof of Lemma 3.6. Write $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ and $v^p(t,x) = m(t+x, t-x)$ with $m(s_1,s_2) = (\eta(s_1) - \eta(s_2))^p$. Define, for $s_2 \le s_1 \le s_2 + 2\pi$,

$$M(s_1,s_2) := -\frac{1}{8}\int_{Q_{s_1,s_2}} m(\xi_1,\xi_2)\, d\xi_1\, d\xi_2, \tag{21}$$

where $Q_{s_1,s_2} := \{(\xi_1,\xi_2) \in \mathbf{R}^2 \mid s_1 \le \xi_1 \le s_2 + 2\pi,\ s_2 \le \xi_2 \le s_1\}$.

The partial derivative of M w.r.t. s_1 is given by

$$\partial_{s_1} M(s_1,s_2) = -\frac{1}{8}\int_{s_1}^{s_2+2\pi} m(\xi_1,s_1)\, d\xi_1 + \frac{1}{8}\int_{s_2}^{s_1} m(s_1,\xi_2)\, d\xi_2.$$

Differentiating w.r.t. s_2, remembering that $m(s_1,s_2) = (\eta(s_1) - \eta(s_2))^p$ with p even, and that η is 2π-periodic, we obtain

$$\partial_{s_2}\partial_{s_1} M = -\frac{1}{8}m(s_2 + 2\pi, s_1) - \frac{1}{8}m(s_1,s_2) = -\frac{1}{4}m(s_1,s_2).$$

Moreover $M(s_1,s_1) = M(s_1, s_1 - 2\pi) = 0$ and $M(s_1 + 2\pi, s_2 + 2\pi) = M(s_1,s_2)$.

This implies that $z(t,x) = M(t+x, t-x)$ satisfies $-z_{tt} + z_{xx} = v^p$ and $z(t,0) = z(t,\pi) = 0$, $z(t+2\pi,x) = z(t,x)$. As a consequence, $z \in L^{-1}(v^p) + V$ and $G(v) = (a^2/2)\int_\Omega v^p z$.

By (21), since (p being even) $m \ge 0$, M and $z \le 0$. Hence $G(v) \le 0$. If $G(v) = 0$, then $v^p(t,x)z(t,x)$ vanishes everywhere, which implies that $m(s_1,s_2) = 0$ or $M(s_1,s_2) = 0$ for all $s_2 \le s_1 \le s_2 + 2\pi$. If $M(s_1,s_2) = 0$, then $m(\xi_1,\xi_2) = 0$ for all $(\xi_1,\xi_2) \in Q_{s_1,s_2}$ and so $m(s_1,s_2) = 0$. In any case $m(s_1,s_2) = 0$, and so $v = 0$. We can conclude that $G(v) < 0$ for all $v \ne 0$. $\square$

Acknowledgements. The authors thank A. Ambrosetti, D. Bambusi and M. Procesi for useful discussions. Part of this paper was written when the second author was visiting S.I.S.S.A. in Trieste.

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Communicated by G. Gallavotti